Geometric quantization of relativistic Hamiltonian mechanics

GENNADI SARDANASHVILY[1]

Department of Theoretical Physics, Physics Faculty, Moscow State University, 117234 Moscow, Russia

Abstract. A relativistic Hamiltonian mechanical system is seen as a conservative Dirac constraint system on the cotangent bundle of a pseudo-Riemannian manifold. We provide geometric quantization of this cotangent bundle where the quantum constraint serves as a relativistic quantum equation.

We are based on the fact that both relativistic and non-relativistic mechanical systems on a configuration space Q can be seen as conservative Dirac constraint systems on the cotangent bundle T^*Q of Q, but occupy its different subbundles. Therefore, one can follow suit of geometric quantization of non-relativistic time-dependent mechanics in order to quantize relativistic mechanics.

Recall that, given a symplectic manifold (Z,Ω) and a Hamiltonian H on Z, a Dirac constraint system on a closed imbedded submanifold $i_N : N \to Z$ of Z is defined as a Hamiltonian system on N provided with the pull-back presymplectic form $\Omega_N = i_N^*\Omega$ and the pull-back Hamiltonian i_N^*H [5, 7, 9]. Its solution is a vector field γ on N which fulfils the equation

$$\gamma\rfloor\Omega_N + i_N^* dH = 0.$$

Let N be coisotropic. Then a solution exists if the Poisson bracket $\{H, f\}$ vanishes on N whenever f is a function vanishing on N. It is the Hamiltonian vector field of H on Z restricted to N.

A configuration space of non-relativistic time-dependent mechanics (henceforth NRM) of m degrees of freedom is an $(m+1)$-dimensional smooth fibre bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$ [7, 11]. It is coordinated by $(q^\lambda) = (q^0, q^i)$, where q^0 is the standard Cartesian coordinate on $\mathbb{R}$. Let T^*Q be the cotangent bundle of Q equipped with the induced coordinates $(q^\lambda, p_\lambda = \dot q_\lambda)$ with respect to the holonomic coframes $\{dq^\lambda\}$. Provided with the canonical symplectic form

$$\Omega = dp_\lambda \wedge dq^\lambda, \tag{1}$$

[1]E-mail address: sard@grav.phys.msu.su; URL: http://webcenter.ru/∼sardan/

the cotangent bundle T^*Q plays the role of a homogeneous momentum phase space of NRM. Its momentum phase space is the vertical cotangent bundle V^*Q of $Q \to \mathbb{R}$ coordinated by (q^λ, q^i). A Hamiltonian $\mathcal{H}$ of NRM is defined as a section $p_0 = -\mathcal{H}$ of the fibre bundle $T^*Q \to V^*Q$. Then the momentum phase space of NRM can be identified with the image N of $\mathcal{H}$ in T^*Q which is the one-codimensional (consequently, coisotropic) imbedded submanifold given by the constraint

$$\mathcal{H}_T = p_0 + \mathcal{H}(q^\lambda, p_k) = 0.$$

Furthermore, a solution of a non-relativistic Hamiltonian system with a Hamiltonian $\mathcal{H}$ is the restriction γ to $N \cong V^*Q$ of the Hamiltonian vector field of $\mathcal{H}_T$ on T^*Q. It obeys the equation $\gamma\rfloor\Omega_N = 0$ [7, 11]. Moreover, one can show that geometric quantization of V^*Q is equivalent to geometric quantization of the cotangent bundle T^*Q where the quantum constraint $\widehat{\mathcal{H}}_T\psi = 0$ on sections ψ of the quantum bundle serves as the Schrödinger equation [3, 4]. This quantization is a variant of quantization of presymplectic manifolds via coisotropic imbeddings [6].

A configuration space of relativistic mechanics (henceforth RM) is an oriented pseudo-Riemannian manifold (Q, g), coordinated by (q^λ). Its momentum phase space is the cotangent bundle T^*Q provided with the symplectic form Ω (1). Note that one also considers another symplectic form $\Omega + F$ where F is the strength of an electromagnetic field [12]. A relativistic Hamiltonian is defined as a smooth real function H on T^*Q [7, 10, 11]. Then a relativistic Hamiltonian system is described as a Dirac constraint system on the subspace N of T^*Q given by the equation

$$H_T = g_{\mu\nu}\partial^\mu H \partial^\nu H - 1 = 0. \tag{2}$$

Similarly to geometric quantization of NRM, we provide geometric quantization of the cotangent bundle T^*Q and characterize a quantum relativistic Hamiltonian system by the quantum constraint

$$\widehat{H}_T\psi = 0. \tag{3}$$

We choose the vertical polarization on T^*Q spanned by the tangent vectors ∂^λ. The corresponding quantum algebra $\mathcal{A} \subset C^\infty(T^*Q)$ consists of affine functions of momenta

$$f = a^\lambda(q^\mu)p_\lambda + b(q^\mu) \tag{4}$$

on T^*Q. They are represented by the Schrödinger operators

$$\widehat{f} = -ia^\lambda\partial_\lambda - \frac{i}{2}\partial_\lambda a^\lambda - \frac{i}{4}a^\lambda\partial_\lambda \ln(-g) + b, \qquad g = \det(g_{\alpha\beta}) \tag{5}$$

in the space $\mathbb{C}^\infty(Q)$ of smooth complex functions on Q.

Note that the function H_T (2) need not belong to the quantum algebra $\mathcal{A}$. Nevertheless, one can show that, if H_T is a polynomial of momenta of degree k, it can be represented as a finite composition

$$H_T = \sum_i f_{1i} \cdots f_{ki} \tag{6}$$

of products of affine functions (4), i.e., as an element of the enveloping algebra $\overline{\mathcal{A}}$ of the Lie algebra $\mathcal{A}$ [3]. Then it is quantized

$$H_T \mapsto \widehat{H}_T = \sum_i \widehat{f}_{1i} \cdots \widehat{f}_{ki} \tag{7}$$

as an element of $\overline{\mathcal{A}}$. However, the representation (6) and, consequently, the quantization (7) fail to be unique.

Let us provide the above mentioned formulation of classical RM as a constraint autonomous mechanics on a pseudo-Riemannian manifold (Q, g) [2, 7, 8]. Note that it need not be a space-time manifold.

The space of relativistic velocities of RM on Q is the the tangent bundle TQ of Q equipped with the induced coordinates $(q^\lambda, \dot q^\lambda)$ with respect to the holonomic frames $\{\partial_\lambda\}$. Relativistic motion is located in the subbundle W_g of hyperboloids

$$g_{\mu\nu}(q)\dot q^\mu \dot q^\nu - 1 = 0 \tag{8}$$

of TQ. It is described by a second order dynamic equation

$$\ddot q^\lambda = \Xi^\lambda(q^\mu, \dot q^\mu) \tag{9}$$

on Q which preserves the subbundle (8), i.e.,

$$(\dot q^\lambda \partial_\lambda + \Xi^\lambda \dot\partial_\lambda)(g_{\mu\nu}\dot q^\mu \dot q^\nu - 1) = 0, \qquad \dot\partial_\lambda = \partial/\partial \dot q^\lambda.$$

This condition holds if the right-hand side of the equation (9) takes the form

$$\Xi^\lambda = \{_\mu{}^\lambda{}_\nu\}\dot q^\mu \dot q^\nu + F^\lambda,$$

where $\{_\mu{}^\lambda{}_\nu\}$ are Cristoffel symbols of a metric g, while F^λ obey the relation $g_{\mu\nu}F^\mu \dot q^\nu = 0$. In particular, if the dynamic equation (9) is a geodesic equation

$$\ddot q^\lambda = K^\lambda_\mu \dot q^\mu$$

with respect to a (non-linear) connection

$$K = dq^\lambda \otimes (\partial_\lambda + K^\mu_\lambda \dot\partial_\mu)$$

on the tangent bundle $TQ \to Q$, this connections splits into the sum

$$K^\lambda_\mu = \{\mu{}^\lambda{}_\nu\}\dot q^\nu + F^\lambda_\mu \tag{10}$$

of the Levi–Civita connection of g and a soldering form

$$F = g^{\lambda\nu}F_{\mu\nu}dq^\mu \otimes \dot\partial_\lambda, \qquad F_{\mu\nu} = -F_{\nu\mu}.$$

As was mentioned above, the momentum phase space of RM on Q is the cotangent bundle T^*Q provided with the symplectic form Ω (1). Let H be a smooth real function on T^*Q such that the morphism

$$\widetilde H : T^*Q \to TQ, \qquad \dot q^\mu = \partial^\mu H \tag{11}$$

is a bundle isomorphism. Then the inverse image $N = \widetilde H^{-1}(W_g)$ of the subbundle of hyperboloids W_g (8) is a one-codimensional (consequently, coisotropic) closed imbedded subbundle of T^*Q given by the constraint $H_T = 0$ (2). We say that H is a relativistic Hamiltonian if the Poisson bracket $\{H, H_T\}$ vanishes on N. This means that the Hamiltonian vector field

$$\gamma = \partial^\lambda H\partial_\lambda - \partial_\lambda H\partial^\lambda \tag{12}$$

of H preserves the constraint N and, restricted to N, it obeys the Hamilton equation

$$\gamma\rfloor\Omega_N + i^*_N dH = 0 \tag{13}$$

of a Dirac constraint system on N with a Hamiltonian H.

The morphism (11) sends the vector field γ (12) onto the vector field

$$\gamma_T = \dot q^\lambda\partial_\lambda + (\partial^\mu H\partial^\lambda\partial_\mu H - \partial_\mu H\partial^\lambda\partial^\mu H)\dot\partial_\lambda$$

on TQ. This vector field defines the second order dynamic equation

$$\ddot q^\lambda = \partial^\mu H\partial^\lambda\partial_\mu H - \partial_\mu H\partial^\lambda\partial^\mu H \tag{14}$$

on Q which preserves the subbundle of hyperboloids (8).

Example 1. The following is a basic example of relativistic Hamiltonian systems. Put

$$H = \frac{1}{2m} g^{\mu\nu}(p_\mu - b_\mu)(p_\nu - b_\nu),$$

where m is a constant and $b_\mu dq^\mu$ is a covector field on Q. Then $H_T = 2m^{-1}H - 1$ and $\{H, H_T\} = 0$. The constraint $H_T = 0$ defines a closed imbedded one-codimensional subbundle N of T^*Q. The Hamilton equation (13) takes the form $\gamma\rfloor\Omega_N = 0$. Its solution (12) reads

$$\dot q^\alpha = \frac{1}{m} g^{\alpha\nu}(p_\nu - b_\nu),$$
$$\dot p_\alpha = -\frac{1}{2m}\partial_\alpha g^{\mu\nu}(p_\mu - b_\mu)(p_\nu - b_\nu) + \frac{1}{m} g^{\mu\nu}(p_\mu - b_\mu)\partial_\alpha b_\nu.$$

The corresponding second order dynamic equation (14) on Q is

$$\ddot q^\lambda = \{_\mu{}^\lambda{}_\nu\}\dot q^\mu \dot q^\nu - \frac{1}{m} g^{\lambda\nu}F_{\mu\nu}\dot q^\mu, \tag{15}$$
$$\{_\mu{}^\lambda{}_\nu\} = -\frac{1}{2} g^{\lambda\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}), \qquad F_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu.$$

It is a geodesic equation with respect to the affine connection

$$K^\lambda_\mu = \{_\mu{}^\lambda{}_\nu\}\dot q^\nu - \frac{1}{m} g^{\lambda\nu}F_{\mu\nu}$$

of type (10). For instance, let g be a metric gravitational field and let $b_\mu = eA_\mu$, where A_μ is an electromagnetic potential whose gauge holds fixed. Then the equation (15) is the well-known equation of motion of a relativistic massive charge in the presence of these fields.

Turn now to quantization of RM. We follow the standard geometric quantization of the cotangent bundle [1, 12, 13]. Because the canonical symplectic form Ω (1) on T^*Q is exact, the prequantum bundle is defined as a trivial complex line bundle C over T^*Q. Note that this bundle need no metaplectic correction since T^*X is endowed with canonical coordinates for the symplectic form Ω. Thus, C is a quantum bundle. Let its trivialization

$$C \cong T^*Q \times \mathbb{C} \tag{16}$$

hold fixed, and let $(q^\lambda, p_\lambda, c)$, $c \in \mathbb{C}$, be the associated bundle coordinates. Then one can treat sections of C (16) as smooth complex functions on T^*Q. Note that another trivialization of C leads to an equivalent quantization of T^*Q.

The Kostant–Souriau prequantization formula associates to each smooth real function $f\in C^\infty(T^*Q)$ on T^*Q the first order differential operator

$$\widehat f = -i\nabla_{\vartheta_f} + f \tag{17}$$

on sections of C, where $\vartheta_f = \partial^\lambda f\partial_\lambda - \partial_\lambda f\partial^\lambda$ is the Hamiltonian vector field of f and ∇ is the covariant differential with respect to a suitable $U(1)$-principal connection A on C. This connection preserves the Hermitian metric $g(c,c')=c\overline{c}'$ on C, and its curvature form obeys the prequantization condition $R=i\Omega$. For the sake of simplicity, let us assume that Q and, consequently, T^*Q is simply connected. Then the connection A up to gauge transformations is

$$A = dp_\lambda\otimes\partial^\lambda + dq^\lambda\otimes(\partial_\lambda + icp_\lambda\partial_c), \tag{18}$$

and the prequantization operators (17) read

$$\widehat f = -i\vartheta_f + (f - p_\lambda\partial^\lambda f). \tag{19}$$

Let us choose the vertical polarization on T^*Q. It is the vertical tangent bundle VT^*Q of the fibration $\pi : T^*Q\to Q$. As was mentioned above, the corresponding quantum algebra $\mathcal{A}\subset C^\infty(T^*Q)$ consists of affine functions f (4) of momenta p_λ. Its representation by operators (19) is defined in the space E of sections ρ of the quantum bundle C of compact support which obey the condition $\nabla_\vartheta\rho = 0$ for any vertical Hamiltonian vector field ϑ on T^*Q. This condition takes the form

$$\partial_\lambda f\partial^\lambda\rho = 0, \qquad \forall f\in C^\infty(Q).$$

It follows that elements of E are independent of momenta and, consequently, fail to be compactly supported, unless $\rho = 0$. This is the well-known problem of Schrödinger quantization which is solved as follows [1, 3].

Let $i_Q : Q\to T^*Q$ be the canonical zero section of the cotangent bundle T^*Q. Let $C_Q = i_Q^*C$ be the pull-back of the bundle C (16) over Q. It is a trivial complex line bundle $C_Q = Q\times\mathbb{C}$ provided with the pull-back Hermitian metric $g(c,c')=c\overline{c}'$ and the pull-back

$$A_Q = i_Q^*A = dq^\lambda\otimes(\partial_\lambda + icp_\lambda\partial_c)$$

of the connection A (18) on C. Sections of C_Q are smooth complex functions on Q, but this bundle need metaplectic correction.

Let the cohomology group $H^2(Q;\mathbb{Z}_2)$ of Q be trivial. Then a metalinear bundle $\mathcal{D}$ of complex half-forms on Q is defined. It admits the canonical lift of any vector field τ on Q such that the corresponding Lie derivative of its sections reads

$$\mathbf{L}_\tau = \tau^\lambda\partial_\lambda + \frac{1}{2}\partial_\lambda\tau^\lambda.$$

Let us consider the tensor product $Y = C_Q \otimes \mathcal{D}$ over Q. Since the Hamiltonian vector fields

$$\vartheta_f = a^\lambda\partial_\lambda - (p_\mu\partial_\lambda a^\mu + \partial_\lambda b)\partial^\lambda$$

of functions f (4) are projected onto Q, one can assign to each element f of the quantum algebra $\mathcal{A}$ the first order differential operator

$$\widehat{f} = (-i\overline{\nabla}_{\pi\vartheta_f} + f)\otimes \mathrm{Id} + \mathrm{Id}\otimes \mathbf{L}_{\pi\vartheta_f} = -ia^\lambda\partial_\lambda - \frac{i}{2}\partial_\lambda a^\lambda + b$$

on sections ρ_Q of Y. For the sake of simplicity, let us choose a trivial metalinear bundle $\mathcal{D}\to Q$ associated to the orientation of Q. Its sections can be written in the form $\rho_Q = (-g)^{1/4}\psi$, where ψ are smooth complex functions on Q. Then the quantum algebra $\mathcal{A}$ can be represented by the operators $\widehat{f}$ (5) in the space $\mathbb{C}^\infty(Q)$ of these functions. It is easily justified that these operators obey the Dirac condition

$$[\widehat{f},\widehat{f}'] = -i\widehat{\{f,f'\}}.$$

Remark 2. One usually considers the subspace $E_Q\subset\mathbb{C}^\infty(Q)$ of functions of compact support. It is a pre-Hilbert space with respect to the non-degenerate Hermitian form

$$\langle\psi|\psi'\rangle = \int_Q \psi\overline{\psi}'(-g)^{1/2}d^{m+1}q$$

It is readily observed that $\widehat{f}$ (5) are symmetric operators $\widehat{f}=\widehat{f}^*$ in E_Q, i.e., $\langle\widehat{f}\psi|\psi'\rangle = \langle\psi|\widehat{f}\psi'\rangle$. In RM, the space E_Q however gets no physical meaning.

As was mentioned above, the function H_T (2) need not belong to the quantum algebra $\mathcal{A}$, but a polynomial function H_T can be quantized as an element of the enveloping algebra $\overline{\mathcal{A}}$ by operators $\widehat{H}_T$ (7). Then the quantum constraint (3) serves as a relativistic quantum equation.

Example 3. Let us consider a massive relativistic charge in Example 1 whose relativistic Hamiltonian is

$$H = \frac{1}{2m} g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu).$$

It defines the constraint

$$H_T = \frac{1}{m^2} g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) - 1 = 0. \tag{20}$$

Let us represent the function H_T (20) as the symmetric product

$$H_T = \frac{(-g)^{-1/4}}{m} \cdot (p_\mu - eA_\mu) \cdot (-g)^{1/4} \cdot g^{\mu\nu} \cdot (-g)^{1/4} \cdot (p_\nu - eA_\nu) \cdot \frac{(-g)^{-1/4}}{m} - 1$$

of affine functions of momenta. It is quantized by the rule (7), where

$$(-g)^{1/4} \circ \widehat{\partial}_\alpha \circ (-g)^{-1/4} = -i\partial_\alpha.$$

Then the well-known relativistic quantum equation

$$(-g)^{-1/2}[(\partial_\mu - ieA_\mu)g^{\mu\nu}(-g)^{1/2}(\partial_\nu - ieA_\nu) + m^2]\psi = 0. \tag{21}$$

is reproduced up to the factor $(-g)^{-1/2}$.

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